MTH 301: Group Theory

Practice Assignment I

1. For any two groups G and H, show that $G \times H \cong H \times G$. [Hint: Show that the map $(g, h) \mapsto (h, g)$ is an isomorphism.]

2. For any three groups G_1, G_2 and G_3,show that

$$G_1 \times (G_2 \times G_3) \cong (G_1 \times G_2) \times G_3$$

.

3. Using the Sylow's Theorems (or otherwise), classify all groups up to isomorphism of orders ≤ 11.

4. Consider the direct product G of k groups G_i of order n_i, for $1 \leq i \leq k$, written as

$$G = \prod_{i=1}^{k} G_i = G_1 \times G_2 \times \ldots \times G_k.$$

   (a) Show that $|G| = n_1 n_2 \ldots n_k$.

   (b) If $g = (g_1, g_2, \ldots, g_k) \in G$, then

$$o(g) = \text{lcm}(o(g_1), o(g_2), \ldots o(g_k)).$$

   [Hint: $o(g)$ is the smallest integer r such that $g_i^r = 1$, for all i.]

   (c) If $G_i = \mathbb{Z}_{m_i}$, for $1 \leq i \leq k$, then show that

$$G \cong \mathbb{Z}_{m_1 m_2 \ldots m_k} \iff \gcd(m_i, m_j) = 1, \text{ for } 1 \leq i < j \leq k.$$

   [Hint: Show this first for the case when $k = 2$ by determining $o((1, 1))$, and then use induction.]

5. Show that if G is a finite p-group, then $|G| = p^k$, for some $k \geq 1$. [Hint: Use Cauchy's Theorem.]